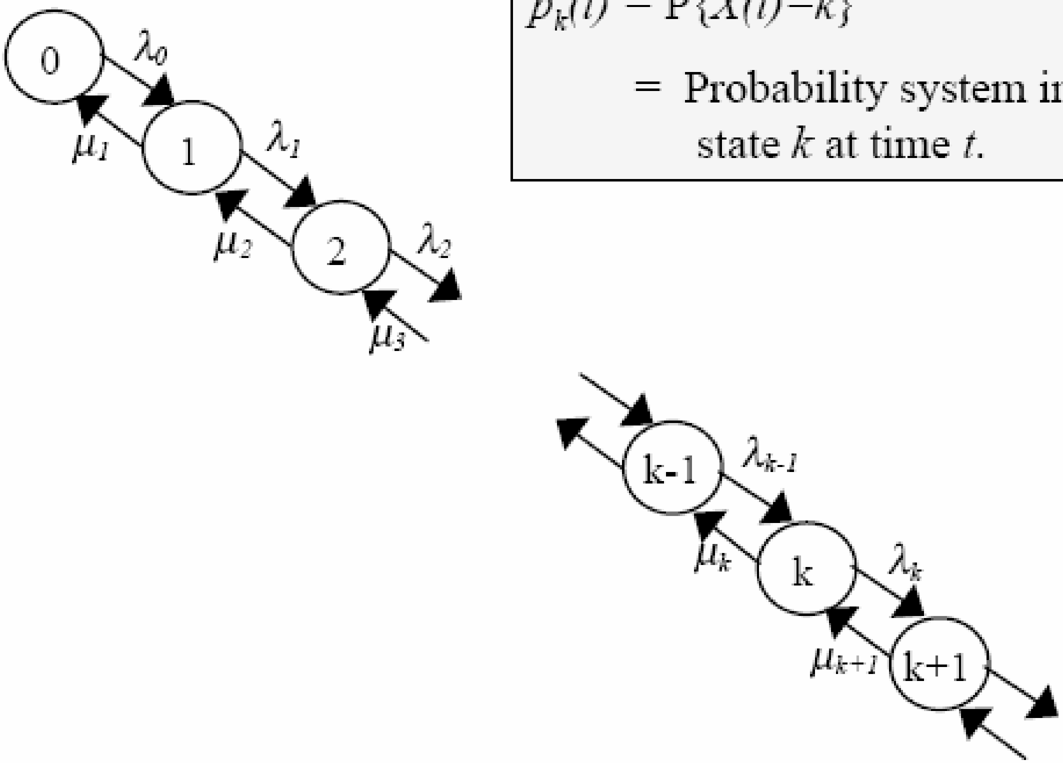
Birth & Death problem

حل مسئله تولد و مرگ به سه روش
و ارائه راه حل زنجیره مارکوف

مدل عمومی سیستم صف با یک خدمت دهنده

$p_k(t) = P\{X(t)=k\}$

$=$ Probability system in state k at time t.



State Transition Diagram for a Birth-Death Process

مدل تولد و مرگ در سیستمهای صف

Let λ_k be the birth rate in state k

μ_k be the death rate in state k

Then
$$\left.\begin{array}{l} P\{\text{state } k \text{ to state } k+1 \text{ in time } \Delta t\} = \lambda_k(\Delta t) \\ P\{\text{state } k \text{ to state } k-1 \text{ in time } \Delta t\} = \mu_k(\Delta t) \\ P\{\text{state } k \text{ to state } k \text{ in time } \Delta t\} = 1 - (\lambda_k + \mu_k)(\Delta t) \\ P\{\text{other transitions in } \Delta t\} = 0 \end{array}\right\} \quad \Delta t \to 0$$

System State $X(t)$ = Number in the system at time t

= Total Births - Total Deaths in $(0, t)$
assuming system starts from state 0 at $t=0$

The initial condition will not matter when we are only interested in the equilibrium state distribution.

مدل حل سیستم صف با یک خدمت دهنده به روش معادلات دیفرانسیل

The state transitions from time t to $t+\Delta t$ will then be governed by the following equations -

$$p_0(t+\Delta t) = p_0(t)[1-\lambda_0 \Delta t] + p_1(t)\mu_1 \Delta t$$

$$p_k(t+\Delta t) = p_k(t)[1-(\lambda_k+\mu_k)\Delta t] + p_{k-1}(t)\lambda_{k-1}\Delta t + p_{k+1}(t)\mu_{k+1}\Delta t$$

$$\sum_{k=0}^{\infty} p_k(t) = 1$$

$\Delta t \rightarrow 0$

$$\left.\begin{aligned} \frac{dp_0(t)}{dt} &= -\lambda_0 p_0(t) + \mu_1 p_1(t) \\ \frac{dp_k(t)}{dt} &= -(\lambda_k+\mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t) \\ \sum_{k=0}^{\infty} p_k(t) &= 1 \end{aligned}\right\} \quad (2.6)$$

Obtain the equilibrium solutions by setting

$$\frac{dp_i(t)}{dt} = 0 \quad \forall i$$

and obtaining the state distribution $p_i \; \forall i$ such that the normalization condition

$$\sum_{i=0}^{\infty} p_i = 1$$

is satisfied.

جواب عمومی مسئله یک خدمت دهنده

This yields the following equations to be solved for the state probabilities under equilibrium conditions-

$$\lambda_0 p_0 = \mu_1 p_1 \qquad k=0$$

$$\lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} = (\lambda_k + \mu_k) p_k \qquad k=1,2,3,\ldots\ldots,$$

$$\sum_{i=0}^{\infty} p_i = 1$$

Product Form Solution

The solution is -

$$p_k = p_0 \left[\prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \right] \qquad (2.7)$$

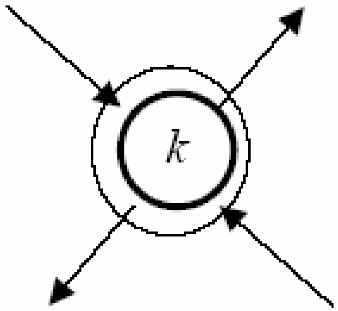
$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (2.8)$$

حل به روش معادلات تعادلی

Balance Equations

حل مسئله به روش معادلات تعادلی سیستم صف

Instead of writing differential equations, one can obtain the solution in a simpler fashion by directly considering *flow balance* for each state.



(a) Draw the state transition diagram

(b) Draw closed boundaries and equate flows across this boundary. Any closed boundary may be chosen for this.

If the closed boundary encloses state k, then we get

Flow entering state $k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} = (\lambda_k+\mu_k)p_k$
= Flow leaving state k

Global Balance Equation for state k

as the desired equation for state k.

(c) Solve the equations in (b) along with the normalization condition to get the equilibrium state distribution.

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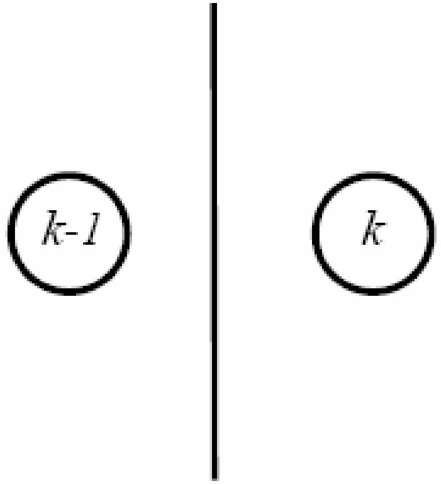
It would be even simpler in this case to consider a closed boundary which is actually closed at infinity



This would lead to the following equation

Flow from state k-1 to k = Flow from state k to k-1

$$\lambda_{k-1}p_{k--1}=\mu_k p_k$$

Detailed Balance Equation

The solution for this will be the same as that obtained earlier

حل مسئله به روش معادلات تعادلی سیستم صف

In general, the equations expressing flow balance in a Birth-Death Chain of this type will be -

$$\sum_{i \neq j} p_i p_{ij} = p_j \sum_{i \neq j} p_{ji}$$

Global Balance Equations

Closed boundary encircling each state *j*

$$p_i p_{ij} = p_j p_{ji}$$

Detailed Balance Equations

Equates flows between states *i* and *j*, in a pair-wise fashion.

Boundary between states *i* and *j*, closed at +∞ and -∞

Markov Chain (MC)

تحلیل صف به روش زنجیره مارکوف

The M/M/1 queue can be modelled by a continuous-time Markov chain with state space $Z_+ = \{0, 1, \ldots\}$ and transition rates

$$\begin{aligned} q(n, n+1) &= \lambda, \quad n \geq 0 \\ q(n, n-1) &= \mu, \quad n \geq 1. \end{aligned}$$

So the transition matrix is

$$\hat{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & \mu & -(\mu+\lambda) & \lambda & \cdots \\ 0 & 0 & \mu & -(\mu+\lambda) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Quasi birth & Death (QBD)

With a suitable ordering of the states, the transition matrix of a QBD has the block partitioned form

$$Q = \begin{bmatrix} \tilde{Q}_1 & Q_0 & 0 & 0 & \cdots \\ Q_2 & Q_1 & Q_0 & 0 & \cdots \\ 0 & Q_2 & Q_1 & Q_0 & \cdots \\ 0 & 0 & Q_2 & Q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

When the QBD is in state (k, i) we say that it is in level k and phase i.

حل مسئله برای سیستم M/M/1

The stationary distribution for a stable M/M/1 queue is derived by solving the system of second order difference equations

$$\pi(n)(\lambda+\mu)=\pi(n-1)\lambda+\pi(n+1)\mu,$$

for $n \geq 1$ and

$$\pi(0)\lambda=\pi(1)\mu.$$

The characteristic equation is

$$x^2\mu - x(\lambda+\mu)+\lambda,$$

which has roots 1 and $\rho=\lambda/\mu$. Thus a summable solution for $\pi(n)$ exists only if $\rho<1$, in which case

$$\pi(n)=(1-\rho)\rho^n.$$

راه حل عمومی برای تعیین احتمال هر حالت

Now let's think about the QBD. Assume it is positive recurrent and write the stationary distribution as $\pi = (\pi_0, \pi_1, \ldots)$.
Then the equations for the stationary distribution are

$$\pi_{n-1}Q_0 + \pi_n Q_1 + \pi_{n+1} Q_2 = \mathbf{0}$$

for $n \geq 1$ and

$$\pi_0 \tilde{Q}_1 + \pi_1 Q_2 = \mathbf{0}.$$

If we happen to be able to find a nonnegative matrix R and a positive vector x_0 such that

$$Q_0 + RQ_1 + R^2 Q_2 = 0,$$

$$x_0 \left[\tilde{Q}_1 + RQ_2\right] = \mathbf{0},$$

and

$$x_0 \sum_{k=0}^{\infty} R^k e = x_0 [I - R]^{-1} e = 1,$$

then the stationary distribution would be given by

$$\pi_n = x_0 R^n.$$

Processes with transition matrices of the form

$$Q = \begin{bmatrix} \tilde{Q}_1 & Q_0 & 0 & 0 & \cdots \\ \tilde{Q}_2 & Q_1 & Q_0 & 0 & \cdots \\ \tilde{Q}_3 & Q_2 & Q_1 & Q_0 & \cdots \\ \tilde{Q}_4 & Q_3 & Q_2 & Q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

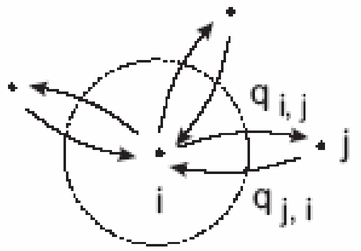
are called Markov chains of GI/M/1 type.

Processes with transition matrices of the form

$$Q = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 & \tilde{Q}_4 & \cdots \\ Q_0 & Q_1 & Q_2 & Q_3 & \cdots \\ 0 & Q_0 & Q_1 & Q_2 & \cdots \\ 0 & 0 & Q_0 & Q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

are called Markov chains of M/G/1 type.

حل مسئله برای فرآیند تولد و مرگ – شکل عمومی



$n+1$ states

$$\boxed{\underbrace{\sum_{j \neq i} \pi_j q_{j,i}}_{\text{flow to state } i} = \underbrace{\sum_{j \neq i} \pi_i q_{i,j}}_{\text{flow out of state } i}}$$

$i = 0, 1, \ldots, n$

one equation per each state

$$\overbrace{(\pi_0, \ldots, \pi_n)}^{\pi} \overbrace{\begin{pmatrix} -\sum_j q_{0,j} & q_{0,1} & q_{0,2} & \cdots & q_{0,n} \\ q_{1,0} & -\sum_j q_{1,j} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,0} & q_{2,1} & -\sum_j q_{2,j} & \cdots & q_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,0} & q_{n,1}, & q_{n,2}, & \cdots & -\sum_j q_{n,j} \end{pmatrix}}^{\mathbf{Q}} = \begin{pmatrix} 0 \\ 0 \\ \\ \vdots \\ 0 \end{pmatrix}$$

$$\boxed{\pi \cdot \mathbf{Q} = \mathbf{0}}$$

one equation is redundant

$$\boxed{\pi_0 + \pi_1 + \cdots + \pi_n = 1}$$

normalization condition

مثال برای حل مسئله زنجیره مارکوف

خصوصیات زنجیره مارکوف

- "**Future**" is independent of "**Past**" given "**Present**"
- In other words: Memoryless
- We've seen memoryless distributions: Exponential and Geometric
- Useful for modeling and analyzing real systems

- A sequence of random variables $\{X_n\}$ is called a ***Markov chain*** if it has the ***Markov property***:

$$\Pr\{X_k = i | X_{k-1}, X_{k-2}, \ldots, X_1\} = \Pr\{X_k = i | X_{k-1}\}$$

- States are usually labeled $\{(0,)1,2,\ldots\}$
- State space can be finite or infinite

Define $\pi_k(i) = \Pr\{X_k = i\}$

Then $\pi_{k+1} = \pi_k P$ (π is a row vector)

Stationary Distribution: $\pi = \lim_{k\to\infty} \pi_k$

if the limit exists.

If π exists, we can solve it by

$$\pi = \pi P, \quad \sum_i \pi(i) = 1$$

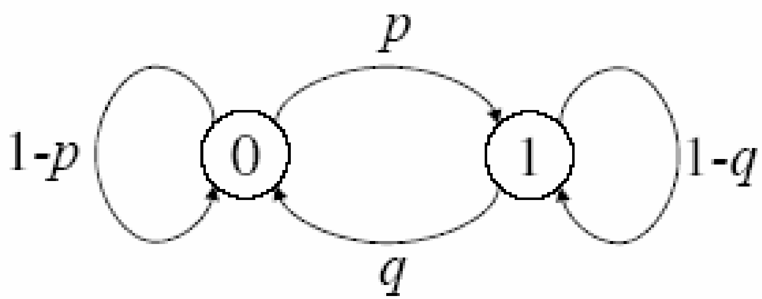
معادلات تعادلی

$$\pi = \pi \begin{pmatrix} 1-\sum_{i\neq 1} p_{1i} & p_{12} & p_{13} & \cdots \\ p_{21} & 1-\sum_{i\neq 2} p_{2i} & p_{23} & \cdots \\ p_{31} & p_{32} & 1-\sum_{i\neq 3} p_{3i} & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

$$\Rightarrow \quad \pi(i) \sum_{j\neq i} p_{ij} = \sum_{j\neq i} \pi(j) p_{ji}$$

- These are called **balance equations**
  - Transitions in and out of state *i* are balanced

حل مسئله برای زنجیره مارکوف دو حالته



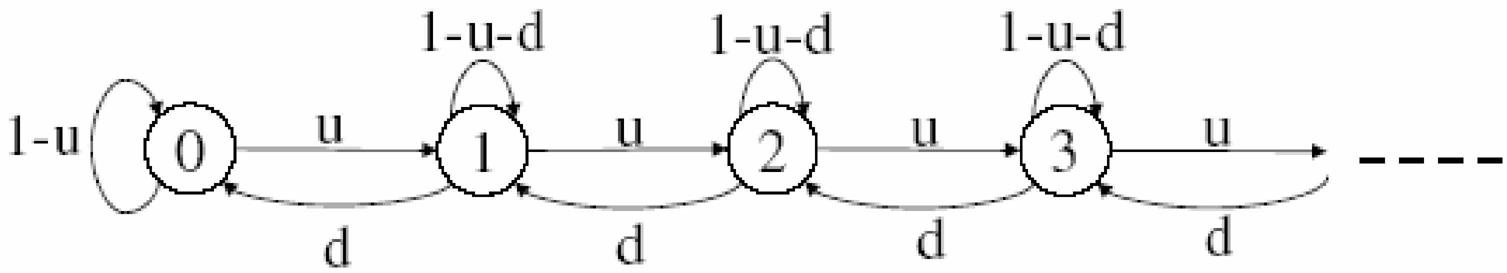
$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

$$(\pi(0) \;\; \pi(1)) = (\pi(0) \;\; \pi(1)) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

$$\Rightarrow \pi(0)p = \pi(1)q$$

$$\pi(0) + \pi(1) = 1 \Rightarrow \pi(0) = \frac{q}{p+q}, \quad \pi(1) = \frac{q}{p+q}$$

حل مسئله برای فرآیند تولد و مرگ



- Arrival w.p. p ; departure w.p. q
- Let u = p(1-q), d = q(1-p), ρ = u/d
- Balance equations:

$$\pi(0)u = \pi(1)d$$

$$\pi(1)(u+d) = \pi(0)u + \pi(2)d$$

$$\Rightarrow \quad \pi(1)u = \pi(2)d$$

$$1 = \sum_{i=0}^{\infty} \pi(i) = \pi(0)(1+\rho+\rho^2+\cdots) = \pi(0)\frac{1}{1-\rho}$$

$$\begin{aligned}
\Rightarrow \quad \pi(0) &= 1-\rho \\
\Rightarrow \quad \pi(i) &= (1-\rho)\rho^i \\
\mathbf{E}(Q) &= \sum_{i=0}^{\infty} i\pi(i) \\
&= (1-\rho)\sum_{i=0}^{\infty} i\rho^i \\
&= \frac{\rho}{1-\rho}
\end{aligned}$$